RAYLEIGH WAVES AND RESONANCE PHENOMENA IN ELASTIC BODIES

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It is characteristic of stationary dynamic problems of elastic theory, to have qualitative changes when the velocity of motion of the distrurbing source becomes equal to or exceeds the velocity of propagation of Rayleigh surface waves.

As an example, we shall consider the plane problem of the wedging of an infinite elastic body by a thin semi-infinite wedge [1]. A crack is formed in front of the wedge. Its surface is stress free everywhere, except in a small end region, where cohesive forces act between opposite boundaries. The cohesive forces ensure a smooth closing of crack boundaries and finite stresses at its end. The length of the crack depends on the velocity of the wedge and its shape, as well as on the material properties. It was shown [1], that as the wedge velocity approaches the Rayleigh velocity, the crack length tends to zero and the stresses in the body be come infinite. It follows, that no crack can form with wedging at a velocity higher than Rayleigh's.

Other examples are the stationary problems of moving loads and dies along the boundary of a halfspace.

The investigation described in [1] of available solutions [2-7] showed, that if the velocity of the load or die approaches the Rayleigh velocity, the stresses and displacements at all points in the body become infinite. This was also observed in [8]. With the transition above Rayleigh's velocity there is a change of sign in the stresses and displacements. In the stationary problem of moving loads, this leads, in particular, to a very unusual change in the form of the free surface. With velocities above Rayleigh's, the material, under a compressive load, is found to swell.

In the present paper an explanation is offered of the above-mentioned resonance phenomena in stationary contact problems. Thus, we consider first the nonstationary problem of the half-plane on the surface of which, beginning at some time, a load, distributed along the semi-infinite side of the boundary, moves uniformly with a velocity below that of sound.

In any bounded region of the space, moving together with the front end of the loading, the solution of this problem tends, with time, to the corresponding stationary solution, and enables it to be investigated.

In the following, we consider a homogeneous, isotropic elastic medium under conditions of plane deformation.

1. Consider an elastic half-plane $y \leqslant 0$, free of stress. At a time $t = 0$, let a normal compressive load of uniform intensity q be applied to the segment $x \leqslant 0$ of the boundary $y = 0$. This load begins to move with constant velocity V in the positive x direction. The load velocity is considered to be lower than the velocity of the slow, transverse, sound waves.

Thus, we seek a solution of the dynamic equations of the theory of elasticity with zero initial and the following boundary conditions ($H(\xi)$ is Heavyside's function):

$$\sigma_y = -qH(Vt - x), \qquad \tau_{xy} = 0 \qquad \text{for} \quad y = 0 \tag{1.1}$$

The formulated problem can be solved by the method used in [9, 10].

It is known (see, for example, [3,6]) that the equations of the dynamic elasticity theory are satisfied, if the stress components σ_y, σ_x and τ_{xy} and displacements u and v are equal to

$$\begin{gathered} u = -\frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x}, \quad \sigma_y = -\rho c_1^2 \Delta\varphi - 2\rho c_2^2\left(\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\varphi}{\partial x^2}\right) \\ \sigma_x = -\rho c_1^2 \triangle\varphi + 2\rho c_2^2\left(\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial^2\varphi}{\partial y^2}\right), \quad \tau_{xy} = -\rho c_2^2\left(2\frac{\partial^2\varphi}{\partial x\partial y} + \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right) \end{gathered} \tag{1.2}$$

Here ϕ, and ψ are the scalar and vector potentials, which are the solutions of the wave equations

$$\triangle\varphi = \frac{1}{c_1^2}\frac{\partial^2\varphi}{\partial t^2}, \qquad \triangle\psi = \frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2}$$

Here ρ is the material density and c_1 and c_2 are the velocities of longitudinal and transverse waves, respectively.

A Laplace time transformation of the wave equations for the potentials and the boundary conditions, using the zero initial conditions, leads to

$$\begin{gathered} \triangle\Phi = \frac{p^2\Phi}{c_1^2}, \qquad \triangle\Psi = \frac{p^2\Psi}{c_2^2} \\ T_{xy} = 0, \qquad \Sigma_y = -\frac{q}{p}\exp\frac{-pH(x)}{V} \qquad \text{for} \quad y = 0 \end{gathered} \tag{1.3}$$

Here Φ, Ψ, Σ and T denote the Laplace representation of the corresponding quantities and p is the parameter of the transformation.

As in [9, 10], we seek the solution of (1.3) in the form

$$\Phi = \int_{-\infty}^{\infty} P(\zeta) \exp\left(ip\zeta x + p\sqrt{\zeta^2 + c_1^{-2}}\, y\right) d\zeta$$
$$\Psi = \int_{-\infty}^{\infty} Q(\zeta) \exp\left(ip\zeta x + p\sqrt{\zeta^2 + c_2^{-2}}\, y\right) d\zeta \qquad (1.4)$$

Here the integration is performed on the real axis in the plane ζ, and $P(\zeta)$ and $Q(\zeta)$ are functions found from the boundary conditions (1.3).

We construct cuts, in the ζ plane, along the imaginary axis from point $(ic_{1,2}^{-1})$ and $(-ic_{1,2}^{-1})$ to $(+i\infty)$ and $(-i\infty)$, respectively, and determine the branches of the $(\zeta^2 + c_{1,2}^{-2})^{1/2}$ roots using the condition $\sqrt{1} = 1$. Also the exponential multipliers in integrals (1.4) will die out for $y \neq 0$.

From the representation (1.4) and relations (1.2) and (1.3), we have the following integral equations for $P(\zeta)$ and $Q(\zeta)$

$$\int_{-\infty}^{\infty} \left[P(\zeta)\, 2i\zeta \sqrt{\zeta^2 + c_1^{-2}} - Q(\zeta)(2\zeta^2 + c_2^{-2})\right] \exp(ip\zeta x)\, d\zeta = 0 \qquad (1.5)$$

$$\int_{-\infty}^{\infty} \left[P(\zeta)(2\zeta^2 + c_2^{-2}) + Q(\zeta)\, 2i\zeta \sqrt{\zeta^2 + c_2^{-2}}\right] \exp(ip\zeta x)\, d\zeta =$$
$$= \frac{K}{p^3} \exp \frac{-pxH(x)}{V} \qquad \left(K = \frac{q}{\rho c_2^2}\right) \qquad (1.6)$$

Assuming

$$P(\zeta) = (2\zeta^2 + c_2^{-2}) R(\zeta), \qquad Q(\zeta) = 2i\zeta \sqrt{\zeta^2 + c_1^{-2}}\, R(\zeta) \qquad (1.7)$$

we satisfy equation (1.5) identically, and from (1.6) we get an equation for the determination of the new unknown function $R(\zeta)$

$$\int_{-\infty}^{\infty} R(\zeta) F_R(\zeta) \exp(ip\zeta x)\, d\zeta = \frac{K}{p^3} \exp \frac{-pxH(x)}{V} \qquad (1.8)$$

Here

$$F_R(\zeta) = (2\zeta^2 + c_2^{-2})^2 - 4\zeta^2 \sqrt{(\zeta^2 + c_1^{-2})(\zeta^2 + c_2^{-2})}$$

is Rayleigh's function. Its only zeros will be $\zeta = \pm iV_R^{-1}$, where V_R is the velocity of the surface Rayleigh waves.

Closing the contour of integration in (1.8) for $x > 0$ and $x < 0$ in the upper and lower halves of the ζ plane, respectively, we note that for $x > 0$ the equation is satisfied if

$$R(\zeta)\, F_R(\zeta) = \frac{KL_+(\zeta)}{2\pi i p^3 L_+(iV^{-1})(\zeta - iV^{-1})} \tag{1.9}$$

and for $x < 0$ if

$$R(\zeta)\, F_R(\zeta) = -\frac{KL_-(\zeta)}{2\pi i p^3 L_-(-i\varepsilon)(\zeta + i\varepsilon)} \qquad (\varepsilon \to 0) \tag{1.10}$$

Here $L_+(\zeta)$ and $L_-(\zeta)$ are analytic functions without any zeros or singularities in the upper and lower half-plane, respectively. On the real axis we have

$$\chi_+(\xi) = \frac{L_+(\xi)}{L_+(iV^{-1})}(\xi + i\varepsilon) = -\frac{L_-(\xi)}{L_-(-i\varepsilon)}(\xi - iV^{-1}) = \chi_-(\xi) \tag{1.11}$$

Thus, $\chi_+(\zeta)$ is the analytic continuation of $\chi_-(\zeta)$ in the upper half-plane. Therefore $\chi_-(\zeta)$ is a function analytic at every point of the finite plane, i.e., an entire function. For the convergence of the integrals (1.4), at $y = 0$, representing Φ and Ψ, we need $\chi_-(\zeta) = A\zeta + B$. Constants A and B are determined from the known values $\chi_-(-i\varepsilon)$ and $\chi_-(iV^{-1})$. Finally we have $\chi_-(\zeta) = i(V^{-1} + \varepsilon)$ and

$$P(\zeta) = \frac{K_1(2\zeta^2 + c_2^{-2})}{2\pi p^3(\zeta + i\varepsilon)(\zeta - iV^{-1})F_R(\zeta)}, \quad Q(\zeta) = \frac{iK_1\zeta\sqrt{\zeta^2 + c_1^{-2}}}{\pi p^3(\zeta + i\varepsilon)(\zeta - iV^{-1})F_R(\zeta)} \tag{1.12}$$

where $K_1 = (K/V)$. From (1.12) it follows that $P(\zeta) \sim \zeta^{-2}$, and $Q(\zeta) \sim \zeta^{-2}$ when $|\zeta| \to \infty$, i.e., the integrals (1.4) converge everywhere, except at the point $x = 0$, $y = 0$.

Substituting the above expressions for $P(\zeta)$ and $Q(\zeta)$ into (1.4) and using the inversion theorem, we find the potentials ϕ and ψ.

In the following we will need the expressions for $x > 0$ of the vertical displacement v and one of the stress components, for example σ_x. The required relations are

$$v = \int_0^t\int_0^\tau v_1(u)\, du\, d\tau \qquad (v_1(u) = J_1(u) + J_2(u) + J_3(u)) \tag{1.13}$$

$$\sigma_x = \int_0^t \sigma_x'\, du \tag{1.14}$$

Here

$$J_1 = H(uc_1 - r)\left[N_1(\zeta_-)\frac{\partial\zeta_-}{\partial u} - N_1(\zeta_+)\frac{\partial\zeta_+}{\partial u}\right]$$

$$J_2 = H(uc_2 - r)\left[N_2(\eta_-)\frac{\partial\eta_-}{\partial u} - N_2(\eta_+)\frac{\partial\eta_+}{\partial u}\right]$$

$$J_3 = f(u)\, g\left(\frac{x}{r}\right)[N_2(\xi_+) - N_2(\xi_-)]\frac{\partial\xi_+}{\partial u}$$

$$\zeta_\pm = \pm\frac{y}{r}\left(\frac{u^2}{r^2} - \frac{1}{c_1^2}\right)^{1/2} + i\frac{ux}{r^2}, \qquad \eta_\pm = \pm\frac{y}{r}\left(\frac{u^2}{r^2} - \frac{1}{c_2^2}\right)^{1/2} + i\frac{ux}{r^2}$$

$$\xi_\pm = i\left\{\frac{ux}{r^2} - \frac{y}{r}\left(\frac{1}{c_2^2} - \frac{u^2}{r^2}\right)^{1/2}\right\} \pm \delta, \qquad \delta \to 0, \qquad r^2 = x^2 + y^2$$

$$N_1(\zeta) = -\frac{K_1(2\zeta^2 + c_2^{-2})\sqrt{\zeta^2 + c_1^{-2}}}{2\pi\zeta(\zeta - iV^{-1})F_R(\zeta)}, \qquad N_2(\eta) = \frac{K_1\eta\sqrt{\eta^2 + c_1^{-2}}}{\pi(\eta - iV^{-1})F_R(\eta)}$$

$$f(u) = 1 \quad \text{for} \quad c_1^{-1}x - y\sqrt{c_2^{-2} - c_1^{-2}} \leqslant u \leqslant c_2^{-1}r, \qquad f(u) = 0 \quad \text{for other } u$$

$$g\left(\frac{x}{r}\right) = 1 \quad \text{for} \quad \frac{c_2}{c_1} \leqslant \frac{x}{r} \leqslant 1, \qquad g\left(\frac{x}{r}\right) = 0 \quad \text{for} \quad \frac{x}{r} < \frac{c_2}{c_1}$$

The expression for σ_x' in equation (1.14), for σ_x, is obtained from the equation of $v_1(u)$ by substituting the functions N_1 and N_2 with functions of the same parameters M_1 and M_2, where

$$M_1(\zeta) = \frac{q(2\zeta^2 + c_2^{-2})(2\zeta^2 + 2c_1^{-2} - c_2^{-2})}{2\pi V\zeta(\zeta - iV^{-1})F_R(\zeta)}$$
$$M_2(\eta) = -\frac{2q\eta\sqrt{(\eta^2 + c_1^{-2})(\eta^2 + c_2^{-2})}}{\pi V(\eta - iV^{-1})F_R(\eta)} \tag{1.15}$$

Equations for the remaining components of the stress and displacement can be written in an analogous fashion.

Next, we shall consider the interrelation of the above solution of the nonstationary problem and the corresponding stationary problem. According to [3, 1], in the stationary solution the stress σ_x at any point of the half-plane is given by

$$\sigma_x = \frac{q}{\pi A}\left[B\left(\frac{\pi}{2} - \text{arc tan}\,\frac{x'}{k_1 y'}\right) - C\left(\frac{\pi}{2} - \text{arc tan}\,\frac{x'}{k_2 y'}\right)\right] \tag{1.16}$$

Here x' and y' determine the position of the point in the system of coordinates connected to the front end of the uniformly moving loading and the coefficients A, B and C are, respectively, equal to

$$A = \sqrt{1 - m^2}\sqrt{1 - \frac{1 - 2\nu}{2(1-\nu)}m^2} - \left(1 - \frac{m^2}{2}\right)^2, \quad B = \left(1 - \frac{m^2}{2}\right)\left(1 + \frac{\nu m^2}{2(1-\nu)}\right)$$
$$C = \sqrt{1 - m^2}\sqrt{1 - \frac{1 - 2\nu}{2(1-\nu)}m^2}, \quad m = \frac{V}{c_2},$$
$$k_1^2 = 1 - \frac{V^2}{c_1^2}, \quad k_2^2 = 1 - \frac{V^2}{c_2^2}$$

For the uniform motion of the loading with the Rayleigh velocity, the coefficient A becomes equal to zero and changes sign as the velocity exceeds the Rayleigh one. This, in particular, means that when $V = V_R$ the stress σ_x becomes infinite at all points in the body. The same happens to all the other components of stress and displacement.

In other words, when the load moves with the Rayleigh velocity, a stationary distribution of stress and displacement in the body is not possible. If, however, the load velocity is different from Rayleigh's, a stationary distribution exists.

2. We shall consider the formulation of the stationary solution in the nonstationary problem, discussed in section 1, of the moving load, distributed along the semi-infinite interval. This will explain the reason for the nonexistence of the stationary solution when the loads move with the Rayleigh velocity. Using the relations of section 1, we shall investigate the change in the stress distribution in an arbitrary fixed neighborhood of the front end of the load. Assume the front end of the load to be the beginning of the moving system of coordinates x' and y': $x = x' + Vt,\ y' = y$.

Consider, for large values of t, the expressions (1.14) and (1.15) for σ_x at an arbitrary point, stationary with respect to the front end of the load (i.e., fixed x' and y'). The form of all three components is similar. We shall consider the derivation of one of them. We have

$$L = \int_{c_1^{-1}r}^{t} \left[M_1(\zeta_-)\frac{\partial\zeta_-}{\partial\tau} - M_1(\zeta_+)\frac{\partial\zeta_+}{\partial\tau}\right]d\tau = \int_{\zeta_+(t)}^{\zeta_-(t)} M_1(\eta)\,d\eta \qquad (2.1)$$

since $\zeta_+(c_1 r) = \zeta_-(c_1 r)$. For large t

$$\zeta_\pm(t) = \pm\frac{y'}{Vt}\sqrt{V^{-2} - c_1^{-2}} + \frac{i}{V}\left(1 - \frac{x'}{Vt}\right) + O\left(\frac{1}{t^2}\right) \qquad (2.2)$$

i.e. $\zeta_\pm(t) \to (i/V)$ as $t \to \infty$. But with $\eta = (i/V)$ the denominator of the function $M_1(\eta)$, under the integral, has either a zero of order one, if $V \neq V_R$, or a zero of order two, when $V = V_R$. Therefore we shall split the integral (2.1) into two, isolating the pole

$$L = L_1 + L_2 = \int_{\zeta_+}^{\zeta_-}\left[M_1(\eta) - \frac{M_1^\circ(iV^{-1})}{(\eta - iV^{-1})F_R(\eta)}\right]d\eta + \int_{\zeta_+}^{\zeta_-}\frac{M_1^\circ(iV^{-1})}{(\eta - iV^{-1})F_R(\eta)}\,d\eta \qquad (2.3)$$

where

$$M_1^\circ(iV^{-1}) = -\frac{iq}{2\pi}(-2V^{-2} + c_2^{-2})(-2V^{-2} + 2c_1^{-2} - c_2^{-2})$$

It can be shown that $L_1 = O(1/t)$. Therefore we only need to calculate L_2.

In the first case, when the load velocity V does not coincide with the Rayleigh velocity V_R, the quantity L_2 has a finite limit. The results of calculating the remaining components of the expression for σ_x are the same. Thus, we obtain that with $V \neq V_R$ the stress σ_x in the vicinity of the front end of the load tends, with time, to a finite limit, agreeing with the known [3, 1] stationary solution (1.16).

In the second case when the load moves with the Rayleigh velocity ($V = V_R$), calculations show that $L_2 \sim \alpha t$ where $\alpha = \alpha(V_R, c_1, c_2, q, x', y')$. Calculating the behavior of the remaining parameters in the expression for σ_x, we get, with coordinates fixed in the moving system

$$\sigma_x = \frac{qy'(-2V_R^{-2}+c_2^{-2})^2\sqrt{V_R^{-2}-c_1^{-2}}}{2\pi\,[8V_R^{-6}(c_1^{-2}-c_2^{-2})-c_2^{-6}(c_2^{-2}-4V_R^{-2})]}\times$$

$$\times\left\{\frac{(-2V_R^{-2}+c_2^{-2})(-2V_R^{-2}+2c_1^{-2}-c_2^{-2})}{(r')^2V_R^{-2}-(y')^2c_1^{-2}}-\frac{4V_R^{-2}(V_R^{-2}-c_2^{-2})}{(r')^2V_R^{-2}-(y')^2c_2^{-2}}\right\}t+O(1) \qquad (2.4)$$

$$((r')^2=(x')^2+(y')^2)$$

It can thus be seen, that the stresses in an arbitrary fixed neighborhood of the front end of the load, moving with the Rayleigh velocity, increase asymptotically proportional to time. In other words, the motion of the medium in the neighborhood of the load end will never become steady. Therefore, the solution of the corresponding stationary problem loses its meaning when the load moves with the Rayleigh velocity.

The increase in the stresses in the neighborhood of the front end of the load, moving with Rayleigh velocity, takes place because, in this case, the energy transmitted by the surface waves accumulates in this region.

Actually, the motion of the semi-infinite loading can be represented as a successive addition to its front end of small overloads, the locations and times of application of which depend on the motion of the loading. Each such overload gives rise, at the instant of its application, to longitudinal, transverse and surface waves. By assumption, the velocity of the front end of the loading is smaller than the velocity of propagation of the transverse, and therefore also of the longitudinal waves. Therefore, with time, the fronts of the longitudinal and transverse waves approach infinity relative to an arbitrary fixed neighborhood of the front end of the load.

With velocities lower or higher than the Rayleigh velocity, the same happens with the surface waves. Finally, in the neighborhood of the front end of the load an equilibrium will be reached between the inflow and outflow of energy.

This picture will change if the load is moving exactly with the Rayleigh velocity. In this case, the surface waves, arising at the front end of the load at different times and propagating in the direction of its motion, will have a common front. This front moves together with the front end of the load. Thus, in the neighborhood of the front end of the load a superposition of surface waves in the same phase occurs. As a result, the energy transmitted by the surface waves accumulates in the vicinity of the front end of the load. This brings about the increase in stress.

The phenomenon described is analogous to resonance in the usual vibrating systems and is caused by the velocity of the disturbing source being equal to the velocity of propagation of the internal waves of the elastic half-plane, i.e., the surface Rayleigh waves.

3. In the stationary solution, as can be seen from (1.16), the stress is of opposite sign for load velocities below and above the Rayleigh velocity [1]. The same happens to the displacements [1]. In particular, the shape of the free surface changes. Contrary to the usual concepts, under a stationary motion of a compressive loading moving with a velocity above Rayleigh's, the material under the load is found to swell. Also, on the other hand, if the loading is tensile, the material contracts.

For an explanation of this phenomenon, we shall again turn to the nonstationary problem. Based on (1.13), we have the following expression for the vertical displacement v of the surface point ($y = 0$, $x > 0$)

$$v = \int_{c_1^{-1}x}^{t} \int_{c_1^{-1}x}^{\tau} [w_1(u) + w_2(u)]\, du d\tau \tag{3.1}$$

$$w_1(u) = -\frac{K}{\pi V c_2^2 u}\left(\frac{u^2}{x^2} - \frac{1}{c_1^2}\right)^{1/2}\left(-2\frac{u^2}{x^2} + \frac{1}{c_2^2}\right)^2\left(\frac{u}{x} - \frac{1}{V}\right)^{-1} \times$$

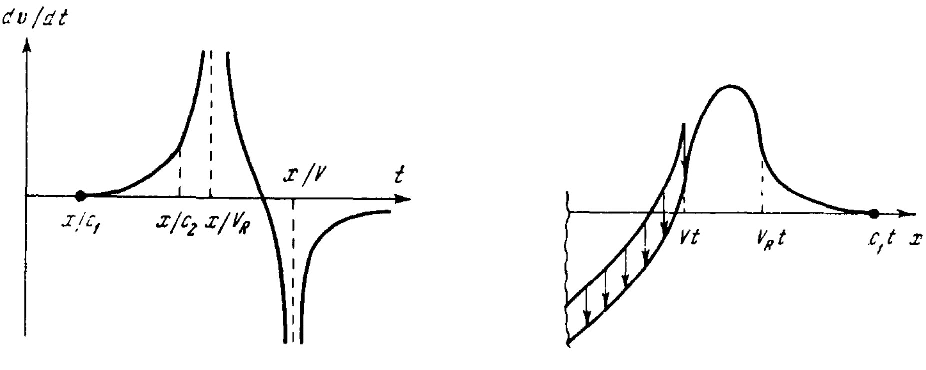
$$\times\left[\left(-2\frac{u^2}{x^2} + \frac{1}{c_2^2}\right)^4 + 16\frac{u^4}{x^4}\left(\frac{u^2}{x^2} - \frac{1}{c_1^2}\right)\left(\frac{1}{c_2^2} - \frac{u^2}{x^2}\right)\right]^{-1} \quad \text{for } c_1^{-1}x \leqslant u \leqslant c_2^{-1}x$$

$$w_1(u) = 0 \quad (u > c_2^{-1}x), \qquad w_2(u) = -\frac{K}{\pi V c_2^2 u}\left(\frac{u^2}{x^2} - \frac{1}{c_1^2}\right)^{1/2}\left(\frac{u}{x} - \frac{1}{V}\right)^{-1} \times$$

$$\times\left[\left(-2\frac{u^2}{x^2} + \frac{1}{c_2^2}\right)^2 - 4\frac{u^2}{x^2}\left(\frac{u^2}{x^2} - \frac{1}{c_1^2}\right)^{1/2}\left(\frac{u^2}{x^2} - \frac{1}{c_2^2}\right)^{1/2}\right]^{-1} \quad \text{for } u \geqslant c_2^{-1}x$$

$$w_2(u) = 0 \quad (u < c_2^{-1}x)$$

Figures 1*a* and 2*a* show the vertical velocity dv/dt at an arbitrary point A ($y = 0$, $x > 0$), as a function of time, when the load is moving with a velocity below and above the Rayleigh velocity, respectively.



a *b*

FIG. 1

From the time of arrival at point A of the longitudinal wave front, $t_1 = c_1^{-1}x$ the velocity begins to increase. Also, for a velocity below Rayleigh's (Fig. 1*a*), a positive infinite velocity is caused by the surface wave, at $t_R = V_R^{-1}x$, originating at the instant the load is applied at its front end. A negative infinite velocity will occur later at point A, at the instant the front end of the loading arrives $(t_0 = V^{-1}x)$.

For a velocity above Rayleigh's the picture is reversed (Fig. 2a) : here the positive infinite velocity is connected with the arrival of the front end of the loading, and the negative infinite velocity with the arrival of the surface wave.

Let us further explain the difference between the cases of velocities below and above the Rayleigh value. In the present problem, the instant of load application is characterized by the fact that, at that instant there is suddenly an infinite load applied to the body, while at subsequent instants there are only small load additions. The motion of the medium can be regarded as a result of the interaction of disturbances which arise as the load is applied to the segment $(-\infty, 0)$ of the x-axis and at subsequent times at the moving front end of the load.

At the initial instant, the particles of the medium situated on the surface near the front end of the load, directly ahead of it, acquire a positive infinite vertical velocity (dv/dt). Similarly, the particles directly behind the front end of the load acquire a negative infinite velocity. Figure 3 shows an example of the velocity distribution in the vicinity of the front end of the load, at a time close to the initial one. Both the positive

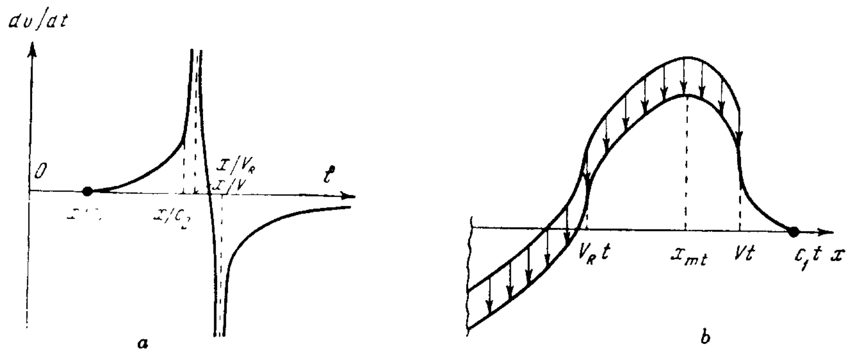


FIG. 2

infinite discontinuity (part 1) of the initial distribution (Fig. 3) and the negative one (part 2) will be carried along the surface wave, which arises at the front end of the load at the instant it is applied. Parts (1) and (2) are situated on opposite sides of the front end of the loading (Fig. 3). They therefore interact in a different manner, for the two cases of lower and higher velocities than Rayleigh's, with the disturbances generated by the front end of the load, as a result of its motion.

Part (1) is situated before the load. For the case of velocities lower than Rayleigh's, according to (3.1), later disturbances do not change its form and velocity of propagation. The surface wave overtakes the front end of the load and brings to point A the corresponding positive infinite discontinuity of the particle vertical velocity (Fig. 1a). Part (2), for velocities lower than Rayleigh's, meets the front end of the load and interacts with it. The result of this interaction is, that the front end of the load, when it reaches point A, brings

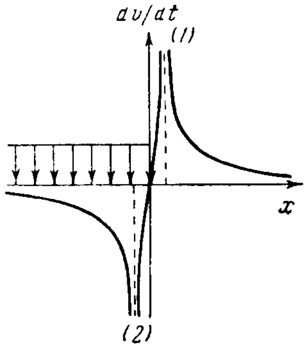


FIG. 3

a change in velocity of the same nature as part (2) of the initial distribution.

For the case of velocities above Rayleigh's, on the other hand, the front end of the load overtakes part (1) of the initial distribution. This leads to a superposition of disturbances, such that, as can be seen from (3.1), the arrival of the front end of the load at any point of the surface will be associated with a positive infinite discontinuity of particle velocity (Fig. 2*a*). The surface wave, however, lags behind the front end of the load and brings the negative infinite discontinuity of velocity, corresponding to part (2) of the initial distribution.

At an arbitrary point A of the surface, the vertical velocity as $t \to \infty$ tends to a constant negative value. Actually, using known properties of Laplace transforms, we have, according to (1.2), (1.4) and (1.12)

$$\lim_{t\to\infty} \frac{dv}{dt} = -\int_{-\infty}^{\infty} \frac{K\sqrt{\zeta^2 + c_1^{-2}}}{2\pi V c_2^2 (\zeta^2 + V^{-2}) F_R(\zeta)} d\zeta$$

The function under the integral dies out at infinity as ζ^{-2}, i.e., the integral converges and is positive since the function under the integral is positive.

The displacement v at any point of the surface dies out to $(-\infty)$ as $t \to \infty$, independent of the load velocity.

In contrast, as can be seen from the above investigation of velocity behavior, the change of displacement at the surface point at the instant the front end of the load and the surface wave arrive depends on the load velocity.

For the case of velocities below Rayleigh's, the displacement at point A increases at the instant the surface wave arrives and decreases when the front end of the load arrives.

For the case of velocities above Rayleigh's, the discontinuity in the vertical velocity, brought about by the front end of the load, corresponds to an increase of displacement at point A, while the discontinuity brought about by the surface wave corresponds to a decrease.

Knowing the vertical velocity (dv/dt) as a function of time (Fig. 1*a* and 2*a*), and, consequently, also the displacement v at any point of the surface, we can construct the distribution of vertical displacements along the surface at any given time.

Figure 1*b* shows the shape of the half-plane surface at some given time, for a velocity below Rayleigh's. The analogous curve, for a velocity above Rayleigh's, is shown in Fig. 2*b*.

To obtain the stationary solution, we must investigate the motion in the vicinity of the front end of the load for large values of time. The larger the time selected, the larger the distance between the points on the surface at which, at that time, we find the front end of the load and the surface wave. In the limit, as $t \to \infty$, this distance becomes infinite.

Thus, for a velocity below Rayleigh's (Fig. 1*b*), the material within an arbitrary neighborhood of the front end of the load will be compressed under the load and will swell in front of it. For a velocity above Rayleigh's, we have swelling under the load and a depression in front of it. These results are in full agreement with those obtained from the stationary solution.

The lift under a compressive load near its front end, (Fig. 2*b*), for a velocity above Rayleigh's, is not connected with the fact that the load is distributed on a semi-infinite interval. If the moving load, with a velocity above Rayleigh's, is applied to a finite length l, its front end will also be lifted relative to the material ahead of the load. Actually, the shape of the surface in this problem is obtained by subtracting from the curve of Fig. 2*b* a similar curve, but moved to the left a distance l. From Fig. 2*b* we can see that the displacement v increases monotonically over the interval (∞, x_{mt}). Here x_{mt} is the coordinate of the surface point at which the displacement is a maximum at the particular time. Therefore, after the motion to the left and subtraction, the displacements v in the above interval will still increase monotonically. For velocities above Rayleigh's, $x_{mt} < Vt$ and, therefore, the front end of the 'finite' load is lifted relative to the material ahead of the load.

Normally, qualitative singularities in stationary problems arise when, in passing through some critical velocity, the type of equation is changed (for example, in passing through the velocity of sound in gas dynamics).

The difference in the investigated resonance phenomena of stationary contact problems with dies or loads moving at Rayleigh's velocity is that they are not connected with changes in the type of equations. These phenomena are caused by boundary conditions: the presence of the free boundary and the appearance of another type of wave, the surface waves. The presence of a free boundary or a surface of separation, along which surface waves can propagate, leads to resonance phenomena in other media also. These effects appear with the motion of the source of disturbance with velocities close to those of surface waves in the given medium. Note that [11] deals with the resonance mechanism of excitation of gravity waves on the surface of a heavy liquid by a turbulent wind. In this case because of dispersion there is not one critical velocity, as in the case of an isotropic homogeneous elastic body, but a spectrum of critical velocities.

It can be expected, that similar phenomena occur in the theory of Cherenkov radiation when a charge moves along a surface of separation between two dielectrics with different dielectric properties.

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